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Differential Equations (MATH 2552)
Mayer

Part 3: Executive Summary

Mystery Circuit

1 Project Topic

Our group will be working on the Mystery Circuit Modelling Scenario from SIMIODE. This applies Kirchhoff's Voltage and Current Laws to the given circuit, which describe, respectively, that the sum of all voltages in a closed loop is zero and the sum of all currents at a node is zero. The circuit we're examining is an RLC (resistor, inductor, capacitor) circuit, with zeroed initial conditions. The specific circuit has two linked loops of resistors and capacitors, in which "gain," the ratio between chosen voltage differentials in the circuit can be modeled mathematically. Because there are two connected loops, there are three different currents. There is the current coming off of the battery $x(t)$, the current split at the middle node becoming $y(t)$ and $z(t)$. We are examining $\frac{E(t)}{z(t)R_{\text{load}}}$ as the "gain" in the system. The first part uses $\omega = 100$ and the entire problem uses $E(t) = \sin(\omega t)$.

2 Progress

From Kirchhoff's Voltage law over the first (xy) loop,

$$E(t) = \sin(\omega t) = x(t)R_1 + \frac{1}{C_1}\int y(t)dt.$$

Kirchhoff's Voltage law also applies to the second (yz) loop:

$$\frac{1}{C_1}\int y(t)dt = \frac{1}{C_2}\int z(t)dt + z(t)R_{\text{load}}.$$

Differentiating and rearranging gives:

$$x'(t) = -\frac{y(t)}{R_1C_1} + \frac{\omega\cos(\omega t)}{R_1},$$

$$z'(t) = \frac{y(t)}{C_1R_{\text{load}}} - \frac{z(t)}{C_2R_{\text{load}}}$$

Kirchhoff's current law tells us that $y(t) + z(t) = x(t)$, so

$$y'(t) = x'(t) - z'(t) = -\frac{y(t)}{R_1C_1} + \frac{\omega\cos(\omega t)}{R_1} - \frac{y(t)}{C_1R_{\text{load}}} + \frac{z(t)}{C_2R_{\text{load}}},$$

giving a system of differential equations to solve.

3 Matrix Representation and Laplace Transform

To determine the relevant properties of the linear system, matrix form is useful (this form was chosen to reduce fractions' usage):

$$bfx' = \frac{1}{R_1C_1C_2R_{\text{load}}}\begin{pmatrix} 0 & -C_2R_{\text{load}} & 0 \\ 0 & -C_2(R_1+R_{\text{load}}) & C_1R_1 \\ 0 & C_2R_1 & -C_1R_1 \end{pmatrix} bfx + \frac{1}{R_1}\begin{pmatrix} \omega\cos(\omega t) \\ \omega\cos(\omega t) \\ 0 \end{pmatrix}.$$

4 Application of Laplace Transformation

We can apply the Laplace Transformation in order to solve this system of differential equations. We have the three equations for x', y', and z', and we can take the Laplace Transform of each of these equations"

$$L\{x' = \frac{-y}{C_1R_1} + \frac{\omega\cos(\omega t)}{R_1}\} \Rightarrow sX(s) - x(0) = \frac{Y(s)}{C_1R_1} + \frac{\omega s}{R_1(s^2+\omega^2)}$$

$$L\{y' = y\frac{-R_1 - R_{\text{load}}}{R_1C_1R_{\text{load}}} + \frac{z}{C_2R_{\text{load}}} + \frac{\omega\cos(\omega t)}{R_1}\}$$

$$\Rightarrow sY(s) - y(0) = Y(s)(\frac{-R_1 - R_{\text{load}}}{R_1C_1R_{\text{load}}}) + \frac{Z(s)}{C_2R_{\text{load}}} + \frac{\omega s}{R_1(s^2+\omega^2)}$$

$$L\{z' = \frac{y}{C_1R_{\text{load}}} - \frac{z}{C_2R_{\text{load}}}\} \Rightarrow sZ(s) - z(0) = \frac{Y(s)}{C_1R_{\text{load}}} - \frac{Z(s)}{C_2R_{\text{load}}}$$

The last two equations we get can be used to solve for $Z(s)$, which we find to be

$$Z(s) = \frac{\omega s(C_1C_2R_{\text{load}}^2)}{(s^2+\omega^2)(s^2+sb+R_{\text{load}})}$$

with $b = C_1R_1R_{\text{load}} + R_1C_2R_{\text{load}} + R_{\text{load}}^2C_2$ to simplify notation.

We can now find the partial fraction decomposition of this[1]:

$$Z(s) = \frac{\omega s(C_1C_2R_{\text{load}}^2)}{(s^2+\omega^2)(s^2+sb+R_{\text{load}})} =$$

$$\frac{As+B}{s^2+\omega^2} + \frac{Cs+D}{s^2+sb+R_{\text{load}}} = \omega s C_1C_2R_{\text{load}}^2$$

We find that

$$A = \frac{C_1C_2R_{\text{load}}^2\omega(R_{\text{load}} - \omega^2)}{b^2\omega^2 + R_{\text{load}}^2 - 2R_{\text{load}} - 2R_{\text{load}}\omega^2 + \omega^2}$$

$$B = \frac{bC_1C_2R_{\text{load}}^2\omega^3}{b^2\omega^2 + R_{\text{load}}^2 - 2R_{\text{load}}\omega^2 + \omega^4}$$

$$C = \frac{-C_1C_2R_{\text{load}}^2\omega(R_{\text{load}} - \omega^2)}{b^2\omega^2 + R_{\text{load}}^2 - 2R_{\text{load}}\omega^2 + \omega^4}$$

$$D = \frac{-bC_1C_1R_{\text{load}}^3\omega}{b^2w^2 + R_{\text{load}}^2 - 2R_{\text{load}}\omega^2 + \omega^4}$$

[1] Wolfram Alpha

The second term of $Z(t)$ can be further simplified into (with r_1 and r_2 as the roots of $s^2 + sb + R_{\text{load}}$, as can be found through the quadratic equation—note that they are complex, but this is handled later) the form

$$\frac{E}{s - r_1} + \frac{F}{s - r_2}.$$

$$E = \frac{D - Cr_1}{r_2 - r_1}$$

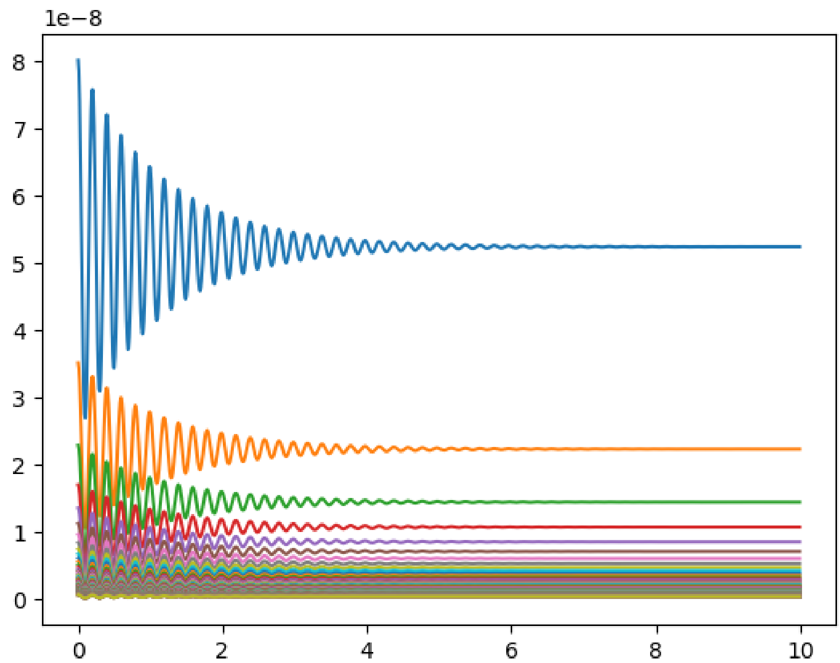
$$F = \overline{E} = C - E.$$

5 Particular Case

There is a particular case which is intended to be investigated:

- $C_1 = 2.5 \times 10^{-6} F$
- $C_2 = 1.0 \times 10^{-6} F$
- $R_1 = 200\Omega$
- $R_{\text{load}} = 1000\Omega$

Using Python with matplotlib (Git), we found that the system converges rapidly towards a steady state with some nearly undetectable oscillation:



This doesn't match up with physical intuition that these varying curves (representing $Z(t)$ with various frequencies of the electromotive force driving the system). However, it does match that because the capacitors absorb some of the variance in current from the source, $Z(t)$ is smaller with smaller values (blue is the smallest frequency at $100Hz$).

However, the initial oscillation in every curve makes a lot of sense from a physical standpoint. With other values for the system, the shapes of these curves vary slightly, mostly in terms of frequency with changes of capacitance and changes of amplitude with changes in the resistance.

6 Possible Generalization

This solution is general to any formulation of the original problem, but gain may look different for a square or triangular wave, for example. It is expected that these would exhibit similar behavior to the sine wave because, because they would input similar amounts of energy on a similar time scale, but the sensitivity to waveform type could be investigated in the same way that this paper did, with possible usage of the Laplace transform to handle discontinuities, but because there is a general form for a sine wave and the output is proportional to the input, a Fourier transform could be used to either approximate or analytically obtain a solution for these types of waves.

7 External Relation

This solution applies the Laplace transform and the inverse Laplace transform, which is directly related to the class as a core component of the class. We also applied, in the first iteration, an attempt at using eigenvalues and eigenvectors to develop a solution was made. This solution would have taken some of the similar paths as this one, especially with the complex roots because all of the eigenvalues were complex. However, the transition to a nonhomogenous system was virtually intractable, which Laplace transforms helped significantly with, especially because they directly accounted for the zero initial current/voltage (even if there were any, these effects would die out quickly after the initialization of the source current).

Outside of this course, this work is likely insufficiently general to provide any real benefit, but the principles discovered, specifically of the gain decreasing with increasing frequencies, are highly applicable to real-life electric systems because if they have resistors or capacitors, they will eventually fall into a similar state if fed by alternating current or direct current as the source. Similar principles *may* be used in the design of a computational system to discover how these effects work on a real circuit, which could have been incorporated into modern circuit design software.